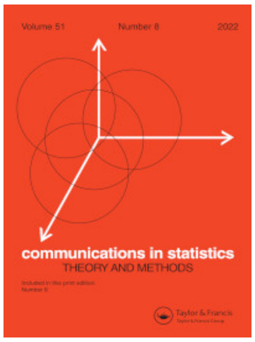

Communications in Statistics - Theory and Methods

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lsta20

Construction of optimal supersaturated designs via generalized Hadamard matrices

Min Li, Min-Qian Liu, Fasheng Sun & Dong Zhang

To cite this article: Min Li, Min-Qian Liu, Fasheng Sun & Dong Zhang (2022) Construction of optimal supersaturated designs via generalized Hadamard matrices, Communications in Statistics - Theory and Methods, 51:8, 2565-2579, DOI: 10.1080/03610926.2020.1777309

To link to this article: https://doi.org/10.1080/03610926.2020.1777309

COMMUNICATIONS IN STATISTICS—THEORY AND METHODS
2022, VOL. 51, NO. 8, 2565–2579
https://doi.org/10.1080/03610926.2020.1777309

Construction of optimal supersaturated designs via generalized Hadamard matrices

Min Li[a], Min-Qian Liu[a], Fasheng Sun[b], and Dong Zhang[c]

[a]School of Statistics and Data Science, LPMC & KLMDASR, Nankai University, Tianjin, China; [b]Department of Statistics, KLAS and School of Mathematics and Statistics, Northeast Normal University, Changchun, China; [c]Mathematics and Digital Sciences, Bloomsburg University, Bloomsburg, Pennsylvania, USA

ABSTRACT

A supersaturated design (SSD) is a factorial design whose run size is not enough for estimating all the main effects. Such designs have received much recent interest because of their potential in factor screening experiments. This paper first shows that the design obtained by the Kronecker sum of a balanced design and a generalized Hadamard matrix (i.e., a matrix with both itself and its transpose being difference matrices) has some nice properties. Based on these findings, some new methods for constructing $E(f_{NOD})$-optimal SSDs via generalized Hadamard matrices are developed. Meanwhile, the non-orthogonality of the proposed designs is well controlled by the source designs. In addition, some generalized Hadamard matrices with nice properties are constructed for obtaining $E(f_{NOD})$-optimal SSDs. The proposed methods are easy to implement and many new SSDs can then be constructed.

ARTICLE HISTORY
Received 26 December 2019
Accepted 28 May 2020

KEYWORDS
Balanced design; coincidence number; difference matrix; non-orthogonality

1. Introduction

A supersaturated design (SSD) is a factorial design in which the number of experimental runs is not large enough for estimating all the main effects. Such designs are used in the initial stages of industrial or scientific experiments for screening the active factors, and are useful when there are a large number of factors under investigation while only a very limited number of runs are available. The designs and their analysis rely on the assumption of effect sparsity, which says that the number of relatively important effects in a factorial experiment is small. The construction of SSDs dates back to Satterthwaite (1959) and Booth and Cox (1962), but such designs were not studied further until the appearance of Lin (1993, 1995) and Wu (1993). Since then, many methods have been proposed for constructing SSDs, including multi-level and mixed-level ones. Readers can refer to Georgiou (2014) for a brief review on the construction of SSDs up until 2012, and refer to Sun, Lin, and Liu (2011), Jones and Majumdar (2014), Xu (2015), Chatterjee et al. (2018), and Jones et al. (2019) for some more recent construction methods. In particular, Sun, Lin, and Liu (2011) introduced a construction based on the

CONTACT Min-Qian Liu mqliu@nankai.edu.cn School of Statistics and Data Science, Nankai University, Tianjin 300071, China.

Kronecker sum of a balanced design and the transpose of a difference matrix. Many optimal SSDs can be obtained by their method. The present paper finds that the SSD obtained by the Kronecker sum of a balanced design and a generalized Hadamard matrix (i.e., a matrix with both itself and its transpose being difference matrices) also has some nice properties. Some new methods for constructing optimal SSDs via generalized Hadamard matrices are developed. In addition, some generalized Hadamard matrices with nice properties are constructed for obtaining optimal SSDs. These are the main contributions of the paper.

In this paper, the optimality of an SSD is measured by the $E(f_{NOD})$ criterion proposed by Fang, Lin, and Liu (2003). Section 2 introduces the $E(f_{NOD})$ criterion. Section 3 shows that the non-orthogonality of the design which is the Kronecker sum of a balanced design and a difference matrix is well controlled by that of the balanced design. Some generalized Hadamard matrices with nice properties are introduced in Section 4, which will be prepared to construct $E(f_{NOD})$-optimal SSDs. The methods for constructing $E(f_{NOD})$-optimal SSDs are proposed in Section 5, along with some discussions on the properties of the resulting designs. Section 6 contains some concluding remarks. All proofs and some large tables are deferred to the Appendix.

2. Optimality criteria for mixed-level SSDs

A mixed-level design that has n runs and m factors with $q_1, ..., q_m$ levels, respectively, is denoted by $D(n; q_1, ..., q_m)$. A $D(n; q_1, ..., q_m)$ can be expressed as an $n \times m$ matrix $D = \left(d_{ij}\right)$. Let d_i be the ith row of D, and d^j be the jth column which takes values from a set of q_j symbols, say $\{g_1, ..., g_{q_j}\}$. If each column d^j has the equal occurrence property of the q_j symbols, we say D is a balanced design. Throughout this paper, we only consider balanced designs. Two columns are called fully aliased if one column can be obtained from the other by permuting levels. When $\sum_{j=1}^{m}(q_j - 1) = n - 1$, the design is saturated. When $\sum_{j=1}^{m}(q_j - 1) > n - 1$, the design is called a supersaturated design (SSD). When some q_j's are equal, we use the notation $D\left(n; q_1^{r_1} \cdots q_l^{r_l}\right)$. And if all the q_j's are equal, the design is said to be symmetrical. A $D(n; q^m)$ is called an orthogonal array (OA) of strength $t \geq 2$, denoted by $OA(n, m, q, t)$, if for any t columns all possible level-combinations appear equally often. Similarly we denote a mixed-level OA of strength t as $OA\left(n, m, q_1^{r_1} \cdots q_l^{r_l}, t\right)$, where $m = r_1 + \cdots + r_l$.

Fang, Lin, and Liu (2003) proposed the $E(f_{NOD})$ criterion for comparing mixed-level SSDs from the viewpoint of orthogonality and uniformity. For any two columns d^i and d^j with levels $g_1, ..., g_{q_i}$ and $h_1, ..., h_{q_j}$, respectively, define

$$f_{NOD}(d^i, d^j) = \sum_{a=1}^{q_i} \sum_{b=1}^{q_j} \left(n_{g_a h_b}^{(ij)} - \frac{n}{q_i q_j} \right)^2 \tag{1}$$

where $n_{g_a h_b}^{(ij)}$ is the number of (g_a, h_b)-pairs in $\left(d^i, d^j\right)$. If $f_{NOD}(d^i, d^j) = 0$, then columns d^i and d^j constitute an $OA(n, 2, q_i q_j, 2)$, in addition, we call them being orthogonal to each other. Then the $E(f_{NOD})$ criterion is defined as minimizing

$$E(f_{NOD}) = \sum_{1 \le i < j \le m} f_{NOD}(d^i, d^j) / \left[m(m-1)/2 \right]$$

To measure the non-orthogonality among all columns of $D(n; q^m)$, Koukouvinos and Mantas (2005) and Chen and Liu (2008) defined

$$f_{\max}^{(D)} = \max\{f_{NOD}(d^i, d^j) | 1 \le i < j \le m\} \tag{2}$$

In addition, Chen and Liu (2008) showed that the upper bound of $f_{NOD}(d^i, d^j)$ is $n^2(q-1)/q^2$, which is achieved by two fully aliased columns. Therefore, for a $D(n; q^m)$, there are no fully aliased columns if $f_{\max}^{(D)} < n^2(q-1)/q^2$. We express this as a lemma for convenience of reference.

Lemma 1. *Let d^1 and d^2 be two balanced columns with n runs and q levels each, if*

$$f_{NOD}(d^1, d^2) < n^2(q-1)/q^2$$

then d^1 and d^2 are not fully aliased.

Besides the $E(f_{NOD})$, there are also several other criteria for evaluating mixed-level SSDs, such as the minimum moment aberration (Xu and Wu 2005), uniformity (Fang, Ge, and Liu 2002b) and χ^2 (Liu, Fang, and Hickernell 2006; Ai, Fang, and He 2007) criteria. All these criteria have close relationships (Liu, Fang, and Hickernell 2006), in particular, the $E(f_{NOD})$ and χ^2 criteria are equivalent for the symmetric case. The $E(f_{NOD})$ criterion is widely used in evaluating and constructing optimal SSDs, see e.g., Liu and Cai (2009), Chatterjee et al. (2018) and the references therein. And in this paper, we mainly use $E(f_{NOD})$ to assess the newly constructed SSDs.

To construct $E(f_{NOD})$ optimal designs, Fang, Lin, and Liu (2003) expressed $E(f_{NOD})$ in terms of the coincidence numbers between distinct rows. The coincidence number between the ith (d_i) and jth (d_j) rows is defined to be the number of k's such that $d_{ik} = d_{jk}$. A design with equal coincidence numbers between distinct rows is called an equidistant design.

Fang et al. (2004b) presented a sufficient condition for a design being $E(f_{NOD})$-optimal.

Lemma 2 (Fang et al. 2004b). *If the difference among all coincidence numbers between distinct rows of design D does not exceed one, then D is $E(f_{NOD})$-optimal.*

In the subsequent sections, we will investigate the coincidence numbers and the non-orthogonality of the proposed designs, and some new $E(f_{NOD})$-optimal designs will then be constructed.

3. Properties of the design obtained via a difference matrix

A difference matrix, denoted by $M_{\mu q, r; q}$, is a $\mu q \times r$ array with entries from a finite Abelian group G with q elements such that every element of G appears exactly μ times in the vector difference between any two columns of the array (Bose and Bush 1952). A matrix M is called a generalized Hadamard matrix if and only if both M and M' are difference matrices (Jungnickel 1979).

For two matrices $B = (b_{ij})$ of order $r \times s$ and C of order $k \times l$, their Kronecker sum is defined to be

Table 1. An $M_{4,4;4}$.

	m^1	m^2	m^3	m^4
1	00	00	00	00
2	00	01	10	11
3	00	10	11	01
4	00	11	01	10

$$B \oplus C = \begin{pmatrix} b_{11}J + C & \cdots & b_{1s}J + C \\ \cdots & \cdots & \cdots \\ b_{r1}J + C & \cdots & b_{rs}J + C \end{pmatrix}$$

where J is a $k \times l$ matrix of ones.

The following result discusses the relationship between the coincidence numbers of a balanced design and those of the design that is the Kronecker sum of this balanced design and the transpose of a difference matrix, which can be obtained from the proof of Theorem 2 of Sun, Lin, and Liu (2011).

Proposition 1. *Let D be a $D(n; q^m)$ and M be an $M_{\mu q, r;q}$, both defined on the same Abelian group G, and Λ be a set consisting of the different values of the coincidence numbers between distinct rows of D, then the coincidence numbers between distinct rows of $M' \oplus_q D$ take values from the set $\Lambda_{\mu q} \cup \{\mu m\}$, where $\Lambda_{\mu q}$ denotes the set with elements being μq times of those elements in Λ, and $\oplus_q$ is the Kronecker sum defined on the Abelian group G.*

Meanwhile, the following theorem shows that the non-orthogonality of the design which is the Kronecker sum of a balanced design and a difference matrix is well controlled by that of the balanced design.

Theorem 1. *Suppose M is an $M_{\mu q, r;q}$ and D is a $D(n; q^m)$ with $f_{\max}^{(D)}$, both defined on a finite Abelian group and the entries are labeled as $G = \{\alpha_0 = 0, \alpha_1, \ldots, \alpha_{q-1}\}$. Define $D_1 = M \oplus_q D$, and let $m^1 \oplus_q d^1$ and $m^2 \oplus_q d^2$ be two different columns of D_1, where m^i and d^i are the columns of M and D respectively, $i = 1, 2$. Then*

$$f_{NOD}\left(m^1 \oplus_q d^1, m^2 \oplus_q d^2\right) = \begin{cases} 0, & m^1 \neq m^2, d^1 = d^2; \qquad (3a) \\ 0, & m^1 \neq m^2, d^1 \neq d^2, \text{ with } m^1 = 0_{\mu q} \text{ or } m^2 = 0_{\mu q}; \qquad (3b) \\ f^*, & \text{otherwise} \qquad (3c) \end{cases}$$

where

$$f^* = \sum_{a \in G} \sum_{b \in G} \left(n_{a-x_1, b-y_1} + n_{a-x_2, b-y_2} + \cdots + n_{a-x_{\mu q}, b-y_{\mu q}}\right)^2 - \mu^2 n^2$$
$$\leq \mu^2 q^2 f_{NOD}(d^1, d^2) = f_{NOD}\left(0_{\mu q} \oplus_q d^1, 0_{\mu q} \oplus_q d^2\right)$$

Thus $f_{\max}^{(D_1)}$ is not greater than $\mu^2 q^2 f_{\max}^{(D)}$, where (x_i, y_i) is the ith row of (m^1, m^2) and $n_{a,b}$ is the number of (a, b)-pairs in (d^1, d^2) for $a, b = \alpha_0, \alpha_1, \ldots, \alpha_{q-1}$.

Now let us see an illustrative example.

Example 1. Suppose M is a generalized Hadamard matrix $M_{4,4;4}$ shown in Table 1, and D is an $E(f_{NOD})$-optimal $D(8; 4^7)$ shown in Table 2, in which the four levels are labeled

Table 2. An $E(f_{NOD})$-optimal $D(8; 4^7)$.

Run	d^1	d^2	d^3	d^4	d^5	d^6	d^7
1	00	11	11	00	10	10	00
2	01	01	00	11	01	10	10
3	11	10	10	00	00	01	10
4	11	00	11	10	01	00	01
5	10	00	01	11	11	01	00
6	01	11	01	01	00	11	01
7	10	10	00	01	10	00	11
8	00	01	10	10	11	11	11

by binary two-dimension vectors, $\{00, 01, 10, 11\}$, the coincidence number between any two distinct rows is 1, and $f_{\max}^{(D)} = f_{NOD}(d^i, d^j) = 4$ for any $1 \leq i \neq j \leq 7$. The design $D_1 = (d_1^1, ..., d_1^{28})$ which is the Kronecker sum of M and D is shown in Table A.1. It is easy to verify that the coincidence number between any two distinct rows of D_1 is 4 or 7. The values of $f_{NOD}(d_1^i, d_1^j)$ are listed in Table A.2, from this table we can observe that $f_{\max}^{(D_1)} = 64$ (far less than 192 which is achieved by two fully aliased columns), $f_{NOD}(d_1^{i+7h}, d_1^{i+7l}) = 0$ for $1 \leq i \leq 7$ and $0 \leq h \neq l < 4$ (c.f, (3a)), and each column from the first 7 columns is orthogonal to any column from the last 21 columns of D_1 (c.f, (3b)). In addition, there are also many other pairs of orthogonal columns in D_1 which can also be seen from Table A.2.

Remark 1.

(i) From Theorem 1, we know that there are no fully aliased columns in the resulting design $D_1 = M \oplus_q D$ if there are no fully aliased columns in D, and the non-orthogonality of D_1 is well controlled by the non-orthogonality of D. Proposition 1 shows that the design generated via the Kronecker sum of a balanced design and the transpose of a difference matrix has a simple structure on the coincidence numbers. Then design $M \oplus_q D$ achieves excellent properties of both Proposition 1 and Theorem 1 if both M and M' are difference matrices (i.e., M is a generalized Hadamard matrix) and D is a balanced design. We will show the construction of generalized Hadamard matrices in next section.

(ii) Besides the columns that are orthogonal to each other guaranteed by (3a) and (3b) of Theorem 1, there may exist many other columns in $M \oplus_q D$ that are also orthogonal to others, see e.g. Table A.2.

(iii) When condition $m^1 = m^2 = 0_{\mu q}$ is not true, the upper bound $\mu^2 q^2 f_{NOD}(d^1, d^2)$ obtained in (3c) (which is usually far less than the one given in Lemma 1) for $f_{NOD}(m^1 \oplus_q d^1, m^2 \oplus_q d^2)$ cannot be reached in most cases, which can also be seen from Table A.2.

4. Construction of generalized Hadamard matrices

Proposition 1 and Theorem 1 show that generalized Hadamard matrices can be used to construct designs with nice properties. Now we present some construction methods for such matrices, which are very useful for the construction of SSDs in the following section.

First, we introduce the concept of Galois Field. The set of residues modulo a prime number p, $\{0, 1, ..., p-1\}$, forms a field of p elements under addition '+' and multiplication modulo p, which is called a Galois field denoted by $GF(p)$. A Galois field of order $q = p^u$ for any prime number p and any positive number u can be obtained as follows. Let $g(x) = b_0 + b_1 x + \cdots + b_u x^u$, with $b_j \in GF(p)$ and $b_u = 1$, be an irreducible polynomial of degree u. Then the set of all polynomials of degree u – 1 or lower, $\{a_0 + a_1 x + \cdots + a_{u-1}x^{u-1} | a_j \in GF(p)\}$, is a Galois field $GF(p^u)$ of order p^u under addition and multiplication of polynomials modulo $g(x)$. For any polynomial $f(x)$ with coefficients from $GF(p)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = q(x)g(x) + r(x)$, with $\deg(r(x)) < u$, where $\deg(r(x))$ is the degree of $r(x)$. $r(x)$ is the residue of $f(x)$ modulo $g(x)$, and we write it as $f(x) = r(x) \pmod{g(x)}$. The multiplicative group $GF(p^u)\backslash\{0\}$ is cyclic, allowing the operation under multiplication easily. Interested readers may refer to the Appendix in Hedayat, Sloane, and Stufken (1999) for more details about Galois field.

We now propose a construction of generalized Hadamard matrix.

Lemma 3. *Let q be a power of an odd prime, and $\alpha_0, \alpha_1, ..., \alpha_{q-1}$ be the elements of GF(q), where $\alpha_0 = 0$ and $\alpha_i = \alpha^i, i = 1, ..., q-1$, for a primitive element α, in particular, $\alpha_{q-1} = \alpha^{q-1} = 1$. Let $\Gamma = (\alpha_0, ..., \alpha_{q-1})', \Phi = (\alpha_0^2, ..., \alpha_{q-1}^2)'$ and 1_q be the $q \times 1$ vector with all elements unity. Then*

(i) $M_1 = \begin{pmatrix} \Gamma\Gamma' & \Gamma\Gamma' + \gamma\Phi 1_q' \\ \Gamma\Gamma' + \beta 1_q \Phi' & \nu\Gamma\Gamma' + \delta 1_q \Phi' + \epsilon \Phi 1_q' \end{pmatrix}$ *is an* $M_{2q, 2q;q}$

where $\beta, \gamma, \delta, \epsilon, \nu$ are any elements of GF(q) that satisfy the conditions

$$\nu \text{ is not a square in } GF(q), \text{ and } \nu = 1 + 4\beta\epsilon = \frac{\epsilon}{\gamma} = \nu^2 - 4\delta\epsilon \tag{4}$$

(ii) *if we take $\nu = \alpha, \beta = 1/2, \gamma = (\alpha-1)/(2\alpha), \delta = \alpha/2, \epsilon = (\alpha-1)/2$ in M_1, and denote the resulting matrix by M, then both M and M′ are $M_{2q,2q;q}$'s, and so M is a generalized Hadamard matrix;*

(iii) *let $D_1 = (d_1^1, ..., d_1^{2q-1})$ be the matrix obtained by omitting the first column of M, then D_1 is a balanced $D(2q; q^{2q-1})$;*

(iv) *$f_{NOD}(d_1^i, d_1^j) = 2q - 4$ for $1 \le i \ne j \le 2q-1$, which is the lower bound of $f_{NOD}(d^i, d^j)$ for any two q-level columns d^i and d^j of $2q (\le q^2)$ runs, and of course there are no fully aliased columns in D_1.*

Let us illustrate the above method using the following example.

Example 2. If we take $\alpha = 2$ as the primitive element of $GF(3)$, then

$$\nu = \alpha = 2, \beta = \frac{1}{2} = 2(\text{since } 2 \times 2 = 4 = 1), \gamma = \frac{\alpha - 1}{2\alpha} = 1, \delta = \frac{\alpha}{2} = 1, \epsilon = \frac{\alpha - 1}{2} = \frac{1}{2} = 2$$

and a difference matrix $M_{6,6;3}$ can be obtained from Lemma 3:

$$M = (0_6, D_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 2 & 1 & 2 \end{pmatrix}$$

where 0_n is the $n \times 1$ vector with all elements zero. Obviously, M is a generalized Hadamard matrix, and $D_1 = (d_1^1, ..., d_1^5)$ is a $D(6; 3^5)$ with $f_{NOD}(d_1^i, d_1^j) = 2$ for any $i \neq j$.

Remark 2. Theorem 6.6 of Hedayat, Sloane, and Stufken (1999) implies that there exists a generalized Hadamard matrix $M_{2q,2q;q}$ for any $q = 2^v$ with $v \geq 1$. And Lemma 3(iv) also holds for $q = 2^v$ with $v \geq 1$.

Since the Kronecker sum of two difference matrices based on a same Abelian group is also a difference matrix (see Lemma 6.38 of Hedayat, Sloane, and Stufken (1999)), then combining Lemma 3 and Remark 2, we get the following result immediately.

Corollary 1. *If q is a prime power, let M be the corresponding generalized Hadamard matrix $M_{2q,2q;q}$ in Lemma 3(ii) or Remark 2, then $\underbrace{M \oplus_q M \oplus_q \cdots \oplus_q M}_{k}$ is a generalized Hadamard matrix $M_{(2q)^k,(2q)^k;q}$ for $k \geq 1$.*

5. Construction of optimal SSDs

This section considers the construction of $E(f_{NOD})$-optimal SSDs. From Lemma 2, we know that the constant coincidence number between any two distinct rows of a design yields $E(f_{NOD})$-optimality. Therefore we present the following theorem which provides a new construction of $E(f_{NOD})$-optimal SSDs.

Theorem 2. *If q is a prime power, let F be the design obtained by deleting the first column of the generalized Hadamard matrix $M_{(2q)^k,(2q)^k;q}$ in Corollary 1, then F is an $E(f_{NOD})$-optimal SSD $D((2q)^k; q^{(2q)^k-1})$ with constant coincidence numbers $\lambda = 2^k q^{k-1} - 1$ and $f_{\max}^{(F)} \leq (2q)^{2(k-1)}(2q-4)$ for $k \geq 1$.*

Example 3 (Example 2 continued). If we delete the first column of M constructed in Example 2, an $E(f_{NOD})$-optimal SSD $D(6; 3^5)$, say F, with $\lambda = 1$ and $f_{\max}^{(F)} = 2$ can be obtained.

$E(f_{NOD})$-optimal $D(n; q^m)$'s with constant coincidence numbers λ constructed by Theorem 2 are summarized in Table 3.

When $k \geq 2$, many column pairs are orthogonal for the designs constructed by Theorem 2. Now we give a corollary for case $k = 2$.

Corollary 2. *Let M be a generalized Hadamard matrix $M_{2q,2q;q}$ generated via Corollary 1, and F be the matrix obtained by deleting the first column of $M \oplus_q M$, then F can be partitioned into*

Table 3. $E(f_{NOD})$-optimal $D(n;q^m)$'s constructed by Theorem 2.

k	n	m	q^*	λ	$f_{\max}^{(F)}$
1	$2q$	$2q-1$	q	1	$2q-4$
2	$4q^2$	$4q^2-1$	q	$4q-1$	$8q^2(q-2)$
k	$(2q)^k$	$(2q)^k-1$	q	$2^kq^{k-1}-1$	$(2q)^{2(k-1)}(2q-4)$

$^*q$ is any prime power and $q \neq 2$; when $q=2$ the resulting designs reduce to OAs.

$$F = (F_1, F_2, \ldots, F_{2q}) \tag{5}$$

where F_1 is a $4q^2 \times (2q-1)$ matrix and F_i is a $4q^2 \times 2q$ matrix for $2 \leq i \leq 2q$. Let G_1 be the new $4q^2 \times (2q-1)$ matrix consisting of the 1st columns of $F_2, \ldots, F_{2q}$, and G_j be the new $4q^2 \times 2q$ matrix consisting of the $(j-1)$th column of F_1 and the jth columns of $F_2, \ldots, F_{2q}$, for $j = 2, \ldots, 2q$. Then

(i) *F_i and G_i are all $OA(4q^2, 2q, q, 2)$'s for $2 \leq i \leq 2q$;*
(ii) *(d^1, F_i) is an $OA(4q^2, 2q+1, q, 2)$ for $2 \leq i \leq 2q$, where d^1 is any column of F_1;*
(iii) *(g^1, G_j) is an $OA(4q^2, 2q+1, q, 2)$ for $2 \leq j \leq 2q$, where g^1 is any column of G_1.*

Example 4. Let M be the generalized Hadamard matrix constructed in Example 2, then according to Theorem 2, we can construct an $E(f_{NOD})$-optimal $D(36; 3^{35})$, i.e., F, with a constant coincidence number 11 between any two distinct rows. This new SSD is listed in Table A.3, where the columns labeled with G_j constitute the matrix G_j for $j = 1, \ldots, 6$. It can also be easily checked that (d^1, F_i) and (g^1, G_j) are $OA(36, 7, 3, 2)$'s for $i, j = 2, \ldots, 6$. And F_i and G_i are all $OA(36, 6, 3, 2)$'s for $i = 2, \ldots, 6$.

Note that the $E(f_{NOD})$-optimal designs that can be constructed through Theorem 2 and Corollary 2 are all symmetrical ones. Now we introduce a method for constructing $E(f_{NOD})$-optimal mixed-level SSDs.

Corollary 3. *Let D_1 be an equidistant design with $2q$ runs, m columns and a constant coincidence number λ between any two distinct rows, where q is a prime power. Suppose $D_2 = (c, D_1)$ with $c = (0, 1, \ldots, 2q-1)'$, and there are no fully aliased columns in D_2. Let $H = (0_{2q} \oplus D_2, F_2, \ldots, F_{2q})$, then*

(i) *the coincidence numbers between distinct rows of H take two values $\lambda + 4q - 2$ or $m + 1 + 2q$;*
(ii) *design (d^*, F_i) is an $OA(4q^2, 2q+1, q^{2q}s^1, 2)$ for $i = 2, \ldots, 2q$, where d^* is any column of $0_{2q} \oplus D_2$ which is supposed to have s levels;*
(iii) *if $|m - \lambda - 2q + 3| \leq 1$, H is an $E(f_{NOD})$-optimal SSD and*

$$f_{\max}^{(H)} \leq 4q^2 \cdot \max\{f_{\max}^{(D_2)}, 2(q-2)\}$$

of course there are no fully aliased columns in H.

Besides the orthogonal columns in Corollary 3(ii), we know that there are many other orthogonal columns in $(F_2, \ldots, F_{2q})$ from Corollary 2.

Example 5. Let D_1 be the equidistant design $D(6; 2^1 3^3)$ with a constant coincidence number 1 between any two distinct rows (c.f., Fang, Lin, and Liu 2003), and $D_2 =$

Table 4. New $E(f_{NOD})$-optimal SSDs H constructed by Corollary 3.

D_1	[Source]*	λ	q	H	$\lambda(H)$ $\lambda+4q-2, m+1+2q$
$D(4;2^3)$	[OA]	1	2	$D(16;2^{15}4^1)$	7, 8
$D(6;3^5)$	[FGL04a]	1	3	$D(36;3^{35}6^1)$	11, 12
$D(6;2^13^3)$	[FLL03]	1	3	$D(36;2^13^{33}6^1)$	11, 11
$D(8;2^7)$	[OA]	3	4	$D(64;2^74^{56}8^1)$	17, 16
$D(8;4^7)$	[FGL02a]	1	4	$D(64;4^{63}8^1)$	15, 16
$D(8;2^14^4)$	[FLL03]	1	4	$D(64;2^14^{60}8^1)$	15, 14
$D(10;5^9)$	[FGL02b]	1	5	$D(100;5^{99}10^1)$	19, 20

*FGL04a: Fang, Ge, and Liu (2004a); FLL03: Fang, Lin, and Liu (2003); FGL02a: Fang, Ge, and Liu (2002a); FGL02b: Fang, Ge, and Liu (2002b).

(c, D_1) with $c = (0, 1, ..., 5)'$, then $H = (0_6 \oplus D_2, D_3)$ is an $E(f_{NOD})$-optimal SSD $D(36; 2^1 3^{33} 6^1)$ with constant coincidence numbers 11, where D_3 consists of the last 30 columns of the $E(f_{NOD})$-optimal $D(36; 3^{35})$ F in Table A.3.

Some new $E(f_{NOD})$-optimal mixed-level SSDs constructed by Corollary 3 are listed in Table 4.

6. Concluding remarks

This paper proposes some methods to construct $E(f_{NOD})$-optimal SSDs, where the generalized Hadamard matrices play a key role in the construction. From Theorem 6.6 of Hedayat, Sloane, and Stufken (1999), we know that, for any prime p and integers $m \geq n \geq 1$, generalized Hadamard matrix $M_{p^m,p^m;p^n}$ exists. In addition, there are many generalized Hadamard matrices on the website http://support.sas.com/techsup/technote/ts723.html maintained by Dr. W.F. Kuhfeld. Thus many $E(f_{NOD})$-optimal SSDs can be obtained. It is important to emphasize that the generalized Hadamard matrices constructed by Corollary 1 have no fully aliased columns and the non-orthogonality can be well controlled. But for the generalized Hadamard matrices from Hedayat, Sloane, and Stufken (1999) or http://support.sas.com/techsup/technote/ts723.html, they may have fully aliased columns. Thus when using a generalized Hadamard matrix that cannot be constructed here to generate $E(f_{NOD})$-optimal SSDs based on the proposed methods, the generalized Hadamard matrix must be chosen carefully.

As for the analysis of SSDs, there are many methods proposed in the literature. In the review paper, Georgiou (2014) provided an informative review on the analysis of SSDs up until 2012. Interested readers can refer to Georgiou (2014) and the the references therein. One can also refer to some recent papers, such as Chen, Weng, and Chu (2013) for a screening procedure using a Bayesian variable selection method, Koukouvinos and Parpoula (2015) for a penalized wrapper method, Balakrishnan, Koukouvinos, and Parpoula (2015) for a method for analyzing data on discrete response regression models, and Drosou and Koukouvinos (2019) for a method based on the support vector machine.

Funding

This work was supported by the National Natural Science Foundation of China (Grant Nos. 11771220 and 11971098), National Ten Thousand Talents Program, Fundamental Research Funds for the Central Universities, Tianjin Development Program for Innovation and

Entrepreneurship, and Tianjin "131" Talents Program. The authorship is listed in alphabetic order.

References

Ai, M. Y., K. T. Fang, and S. Y. He. 2007. $E(\chi^2)$ optimal mixed-level supersaturated designs. *Journal of Statistical Planning and Inference* 137 (1):306–16. doi:10.1016/j.jspi.2005.09.006.

Balakrishnan, N., C. Koukouvinos, and C. Parpoula. 2015. Analyzing supersaturated designs for discrete responses via generalized linear models. *Statistical Papers* 56 (1):121–45. doi:10.1007/s00362-013-0569-z.

Booth, K. H. V., and D. R. Cox. 1962. Some systematic supersaturated designs. *Technometrics* 4 (4):489–95. doi:10.1080/00401706.1962.10490035.

Bose, R. C., and K. A. Bush. 1952. Orthogonal arrays of strength two and three. *The Annals of Mathematical Statistics* 23 (4):508–24. doi:10.1214/aoms/1177729331.

Chatterjee, K., K. Drosou, S. D. Georgiou, and C. Koukouvinos. 2018. Multi-level and mixed-level k-circulant supersaturated designs. *Metrika* 81 (3):337–55. doi:10.1007/s00184-018-0645-4.

Chen, J., and M. Q. Liu. 2008. Optimal mixed-level k-circulant supersaturated designs. *Journal of Statistical Planning and Inference* 138 (12):4151–7. doi:10.1016/j.jspi.2008.03.025.

Chen, R. B., J. Z. Weng, and C. H. Chu. 2013. Screening procedure for supersaturated designs using a Bayesian variable selection method. *Quality and Reliability Engineering International* 29 (1):89–101. doi:10.1002/qre.1299.

Drosou, K., and C. Koukouvinos. 2019. A new variable selection method based on SVM for analyzing supersaturated designs. *Journal of Quality Technology* 51 (1):21–36. doi:10.1080/00224065.2018.1541389.

Fang, K. T., G. N. Ge, and M. Q. Liu. 2002a. Construction of $E(f_{NOD})$-optimal supersaturated designs via Room squares. *Calcutta Statistical Association Bulletin* 52 (1-4):71–84. doi:10.1177/0008068320020503.

Fang, K. T., G. N. Ge, and M. Q. Liu. 2002b. Uniform supersaturated design and its construction. *Science in China Series A: Mathematics* 45 (8):1080–8. doi:10.1007/BF02879992.

Fang, K. T., G. N. Ge, and M. Q. Liu. 2004a. Construction of optimal supersaturated designs by the packing method. *Science in China Series A* 41 (1):128–43. doi:10.1360/02ys0271.

Fang, K. T., G. N. Ge, M. Q. Liu, and H. Qin. 2004b. Combinatorial constructions for optimal supersaturated designs. *Discrete Mathematics.* 279 (1-3):191–202. doi:10.1016/S0012-365X(03)00269-3.

Fang, K. T., D. K. J. Lin, and M. Q. Liu. 2003. Optimal mixed-level supersaturated design. *Metrika* 58 (3):279–91. doi:10.1007/s001840300266.

Georgiou, S. D. 2014. Supersaturated designs: A review of their construction and analysis. *Journal of Statistical Planning and Inference* 144:92–109. doi:10.1016/j.jspi.2012.09.014.

Hedayat, A. S., N. J. A. Sloane, and J. Stufken. 1999. *Orthogonal arrays: Theory and applications*. New York: Springer-Verlag.

Jones, B., R. Lekivetz, D. Majumdar, C. J. Nachtsheim, and J. W. Stallrich. 2019. Construction, properties, and analysis of group-orthogonal supersaturated designs. *Technometrics*. Advance online publication. doi:10.1080/00401706.2019.1654926.

Jones, B., and D. Majumdar. 2014. Optimal supersaturated designs. *Journal of the American Statistical Association* 109 (508):1592–600. doi:10.1080/01621459.2014.938810.

Jungnickel, D. 1979. On difference matrices, resolvable transversal designs and generalized Hadamard matrices. *Mathematische Zeitschrift* 167 (1):49–60. doi:10.1007/BF01215243.

Koukouvinos, C., and P. Mantas. 2005. Construction of some $E(f_{NOD})$ optimal mixed-level supersaturated designs. *Statistics & Probability Letters* 74 (4):312–21. doi:10.1016/j.spl.2005.04.047.

Koukouvinos, C., and C. Parpoula. 2015. A penalized wrapper method for screening main effects and interactions in supersaturated designs. *Quality and Reliability Engineering International* 31 (8):1423–35. doi:10.1002/qre.1679.

Lin, D. K. J. 1993. A new class of supersaturated designs. *Technometrics* 35 (1):28–31. doi:10.1080/00401706.1993.10484990.

Lin, D. K. J. 1995. Generating systematic supersaturated designs. *Technometrics* 37 (2):213–25. doi:10.1080/00401706.1995.10484305.
Liu, M. Q., and Z. Y. Cai. 2009. Construction of mixed-level supersaturated designs by the substitution method. *Statistica Sinica* 19:1705–19.
Liu, M. Q., K. T. Fang, and F. J. Hickernell. 2006. Connections among different criteria for asymmetrical fractional factorial designs. *Statistica Sinica* 16:1285–97.
Satterthwaite, F. 1959. Random balance experimentation (with discussion). *Technometrics* 1 (2): 111–37. doi:10.1080/00401706.1959.10489853.
Sun, F. S., D. K. J. Lin, and M. Q. Liu. 2011. On construction of optimal mixed-level supersaturated designs. *The Annals of Statistics* 39 (2):1310–33. doi:10.1214/11-AOS877.
Wu, C. F. J. 1993. Construction of supersaturated designs through partially aliased interactions. *Biometrika* 80 (3):661–9. doi:10.1093/biomet/80.3.661.
Xu, H. 2015. Nonregular factorial and supersaturated designs. In *Handbook of design and analysis of experiments*, ed. Dean A, Morris M, Stufken J, Bingham D, 339–70. Boca Raton: CRC Press.
Xu, H., and C. F. J. Wu. 2005. Construction of optimal multi-level supersaturated designs. *The Annals of Statistics* 33 (6):2811–36. doi:10.1214/009053605000000688.

Appendix A. Proofs and some large tables

A.1. Proof of Theorem 1

Equation (3a) is well-known for standard construction of OAs from difference matrices, and Equation (3a) is clear from the adding column technique here.

For Equation (3c), let $n^*_{a,b}$ be the number of (a, b)-pairs in $\left(m^1 \oplus_q {d}^1, m^2 \oplus_q d^2\right)$, $n_{a,b}$ be the number of (a, b)-pairs in (d^1, d^2), and

$$(m^1, m^2) = \begin{pmatrix} x_1 & x_2 & \cdots & x_{\mu q} \\ y_1 & y_2 & \cdots & y_{\mu q} \end{pmatrix}'$$

Then,

$$\begin{aligned}
&f_{NOD}(m^1 \oplus d^1, m^2 \oplus d^2) \\
&= \sum_{a\in G}\sum_{b\in G}\left(n^*_{a,b} - \frac{\mu q n}{q^2}\right)^2 = \sum_{a\in G}\sum_{b\in G} n^{*2}_{a,b} - \mu^2 n^2 \\
&= \sum_{a\in G}\sum_{b\in G}(n_{a-x_1,b-y_1} + n_{a-x_2,b-y_2} + \cdots + n_{a-x_{\mu q},b-y_{\mu q}})^2 - \mu^2 n^2 \\
&\le \sum_{a\in G}\sum_{b\in G}\mu q\left(n^2_{a-x_1,b-y_1} + n^2_{a-x_2,b-y_2} + \cdots + n^2_{a-x_{\mu q},b-y_{\mu q}}\right) - \mu^2 n^2 \\
&= \mu q\sum_{a\in G}\sum_{b\in G}\left(n^2_{a-x_1,b-y_1} + n^2_{a-x_2,b-y_2} + \cdots + n^2_{a-x_{\mu q},b-y_{\mu q}}\right) - \mu^2 n^2 \\
&= \mu q\sum_{a\in G}\sum_{b\in G}\mu q n^2_{a,b} - \mu^2 n^2 = \mu^2 q^2\left(\sum_{a\in G}\sum_{b\in G} n^2_{a,b} - \frac{n^2}{q^2}\right) \\
&= \mu^2 q^2 f_{NOD}(d^1, d^2)
\end{aligned}$$

In addition,

$$\begin{aligned}
f_{NOD}\left(0_{\mu q} \oplus_q d^1, 0_{\mu q} \oplus_q d^2\right) &= \sum_{a\in G}\sum_{b\in G}\left(n^*_{a,b} - \frac{\mu q n}{q^2}\right)^2 = \sum_{a\in G}\sum_{b\in G}\left(\mu q n_{a,b} - \frac{\mu q n}{q^2}\right)^2 \\
&= \mu^2 q^2\sum_{a\in G}\sum_{b\in G}\left(n_{a,b} - \frac{n}{q^2}\right)^2 = \mu^2 q^2 f_{NOD}(d^1, d^2)
\end{aligned}$$

Thus we complete the proof.

A.2. Proof of Lemma 3

Part (i) can be obtained from the proof of Theorem 6.33 in Hedayat, Sloane, and Stufken (1999). In addition, part (ii) is also evident from Equation (6.8) of the same book.

Part (iii) follows from the fact that M is a difference matrix with all elements zero in the first column.

For part (iv), note that for D_1, the number of runs $2q$ is less than q^2, we need only to show that there are no duplicate rows in (d_1^i, d_1^j), where d_1^i and d_1^j are any two different columns of D_1, then the value of $f_{NOD}(d_1^i, d_1^j)$ can be directly calculated, and the lower bound can be found in Theorem 1(iii) of Fang, Ge, and Liu (2004a). In fact, suppose the r_1th and r_2th rows in (d_1^i, d_1^j) are the same, then there are at least three zeros in the vector difference between the r_1th and r_2th columns of M', by noting that the elements in the first column of M' are all zeros, this contradicts the fact that M' is an $M_{2q,2q;q}$.

A.3. Proof of Theorem 2

By recursively using the result of Proposition 1 to the generalized Hadamard matrix $\underbrace{M\oplus_q \cdots \oplus_q M}_{k}$ in Corollary 1, and by noting that F is obtained by deleting the first column of this matrix, we can directly obtain that F has a constant coincidence number $\lambda = 2^k q^{k-1} - 1$ between any two distinct rows, and thus it is an $E(f_{NOD})$-optimal SSD (c.f., Lemma 2). Also, it can be easily verified that $f_{\max}^{(F)} \le (2q)^{2(k-1)}(2q-4)$ for $k \ge 1$.

A.4. Proof of Corollary 2

Assertion (i) follows from Lemma 6.12 of Hedayat, Sloane, and Stufken (1999), and (ii) and (iii) follow from Theorem 1.

A.5. Proof of Corollary 3

Write $H = (H_1, H_2) = (0_{2q} \oplus D_2, F_2, \dots, F_{2q})$.

(i) The coincidence number between any two distinct rows of H follows by calculating that of the corresponding rows of H_1 and H_2, respectively.

(ii) Let f^* be any column of F_i, we need only to show that d^* and f^* are orthogonal to each other. In fact this is true by noting that

$$(d^*, f^*) = \begin{pmatrix} d & \alpha_0 \oplus_q f \\ d & \alpha_1 \oplus_q f \\ \vdots & \vdots \\ d & \alpha_{q-1} \oplus_q f \end{pmatrix}$$

which is obviously an OA with two columns, where d and f are some balanced columns.

(iii) The conclusion follows from Lemma 2 and Theorem 2.

A.6. Some large tables

Table A.1. The $D(32; 4^{28})$ obtained via the Kronecker sum of the $D(8; 4^7)$ and $M_{4,4;4}$.

Run	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	00	11	11	00	10	10	00	00	11	11	00	10	10	00	00	11	11	00	10	10	00	00	11	11	00	10	10	00
2	01	01	00	11	01	10	10	01	01	00	11	01	10	10	01	01	00	11	01	10	10	01	01	00	11	01	10	10
3	11	10	10	00	00	01	10	11	10	10	00	00	01	10	11	10	10	00	00	01	10	11	10	10	00	00	01	10
4	11	00	11	10	01	00	01	11	00	11	10	01	00	01	11	00	11	10	01	00	01	11	00	11	10	01	00	01
5	10	00	01	11	11	01	00	10	00	01	11	11	01	00	10	00	01	11	11	01	00	10	00	01	11	11	01	00
6	01	11	01	01	00	11	01	01	11	01	01	00	11	01	01	11	01	01	00	11	01	01	11	01	01	00	11	01
7	10	10	00	01	10	00	11	10	10	00	01	10	00	11	10	10	00	01	10	00	11	10	10	00	01	10	00	11
8	00	01	10	10	11	11	11	00	01	10	10	11	11	11	00	01	10	10	11	11	11	00	01	10	10	11	11	11
9	00	11	11	00	10	10	00	01	10	10	01	11	11	01	10	01	01	10	00	00	10	11	00	00	11	01	01	11
10	01	01	00	11	01	10	10	00	00	01	10	00	11	11	11	11	10	01	11	00	00	10	10	11	00	10	01	01
11	11	10	10	00	00	01	10	10	11	11	01	01	00	11	01	00	00	10	10	11	00	00	01	01	11	11	10	01
12	11	00	11	10	01	00	01	10	01	10	11	00	01	00	01	10	01	00	11	10	11	00	11	00	01	10	11	10
13	10	00	01	11	11	01	00	11	01	00	10	10	00	01	00	10	11	01	01	11	10	01	11	10	00	00	10	11
14	01	11	01	01	00	11	01	00	10	00	00	01	10	00	11	01	11	11	10	01	11	10	00	10	10	11	00	10
15	10	10	00	01	10	00	11	11	11	01	00	11	01	10	00	00	10	11	00	10	01	01	01	11	10	01	11	00
16	00	01	10	10	11	11	11	01	00	11	11	10	10	10	10	11	00	00	01	01	01	11	10	01	01	00	00	00
17	00	11	11	00	10	10	00	10	01	01	10	00	00	10	11	00	00	11	01	01	11	01	10	10	01	11	11	01
18	01	01	00	11	01	10	10	11	11	10	01	11	00	00	10	10	11	00	10	01	01	00	00	01	10	00	11	11
19	11	10	10	00	00	01	10	01	00	00	10	10	11	00	00	01	01	11	11	10	01	10	11	11	01	01	00	11
20	11	00	11	10	01	00	01	01	10	01	00	11	10	11	00	11	00	01	10	11	10	10	01	10	11	00	01	00
21	10	00	01	11	11	01	00	00	10	11	01	01	11	10	01	11	10	00	00	10	11	11	01	00	10	10	00	01
22	01	11	01	01	00	11	01	11	01	11	11	10	01	11	10	00	10	10	11	00	10	00	10	00	00	01	10	00
23	10	10	00	01	10	00	11	00	00	10	11	00	10	01	01	01	11	10	01	11	00	11	11	01	00	11	01	10
24	00	01	10	10	11	11	11	10	11	00	00	01	01	01	11	10	01	01	00	00	00	01	00	11	11	10	10	10
25	00	11	11	00	10	10	00	11	00	00	11	01	01	11	01	10	10	01	11	11	01	10	01	01	10	00	00	10
26	01	01	00	11	01	10	10	10	10	11	00	10	01	01	00	00	01	10	00	11	11	11	11	10	01	11	00	00
27	11	10	10	00	00	01	10	00	01	01	11	11	10	01	10	11	11	01	01	00	11	01	00	00	10	10	11	00
28	11	00	11	10	01	00	01	00	11	00	01	10	11	10	10	01	10	11	00	01	00	01	10	01	00	11	10	11
29	10	00	01	11	11	01	00	01	11	10	00	00	10	11	11	01	00	10	10	00	01	00	10	11	01	01	11	10
30	01	11	01	01	00	11	01	10	00	10	10	11	00	10	00	10	00	00	01	10	00	11	01	11	11	10	01	11
31	10	10	00	01	10	00	11	01	01	11	10	01	11	00	11	11	01	00	11	01	10	00	00	10	11	00	10	01
32	00	01	10	10	11	11	11	11	10	01	01	00	00	00	01	00	11	11	10	10	10	10	11	00	00	01	01	01

Table A.2. The values of $f_{NOD}(d_1^i, d_1^j)$ for the $D(32; 4^{28})$ in Table A.1.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	192	64	64	64	64	64	64	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	64	192	64	64	64	64	64	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	64	64	192	64	64	64	64	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	64	64	64	192	64	64	64	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	64	64	64	64	192	64	64	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	64	64	64	64	64	192	64	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	64	64	64	64	64	64	192	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	192	0	0	0	0	64	0	0	64	64	0	0	0	64	0	0	0	64	64	0	0
9	0	0	0	0	0	0	0	0	192	0	64	0	0	0	0	0	0	0	64	0	64	64	0	64	0	0	64	0
10	0	0	0	0	0	0	0	0	0	192	0	64	0	0	0	64	0	64	0	0	0	64	0	0	0	0	64	64
11	0	0	0	0	0	0	0	0	64	0	192	0	64	0	64	0	0	0	64	0	0	0	0	64	0	0	0	64
12	0	0	0	0	0	0	0	0	0	64	0	192	0	0	64	0	0	0	0	0	0	0	64	0	64	0	64	64
13	0	0	0	0	0	0	0	64	0	0	64	0	192	0	0	64	64	0	64	0	0	0	0	0	0	0	0	64
14	0	0	0	0	0	0	0	0	0	0	0	0	0	192	0	0	64	64	64	64	0	64	64	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	64	64	0	0	192	0	0	0	0	64	0	0	64	64	0	0	0	64
16	0	0	0	0	0	0	0	64	0	64	0	0	64	0	0	192	0	64	0	0	0	0	0	0	0	64	0	64
17	0	0	0	0	0	0	0	64	0	0	0	0	64	64	0	0	192	0	64	0	0	0	64	0	64	0	0	0
18	0	0	0	0	0	0	0	0	0	64	0	0	0	64	0	64	0	192	0	64	0	64	0	0	0	64	0	0
19	0	0	0	0	0	0	0	0	64	0	64	0	64	64	0	0	64	0	192	0	0	64	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	64	64	0	0	64	0	192	0	0	64	64	0	64	0	0
21	0	0	0	0	0	0	0	64	64	0	0	0	0	0	0	0	0	0	0	0	192	0	0	64	64	64	64	0
22	0	0	0	0	0	0	0	0	64	64	0	0	0	64	0	0	0	64	64	0	0	192	0	0	0	0	64	0
23	0	0	0	0	0	0	0	0	0	0	0	64	0	64	64	0	64	0	0	64	0	0	192	0	64	0	0	0
24	0	0	0	0	0	0	0	0	64	0	64	0	0	0	64	0	0	0	0	64	64	0	0	192	0	64	0	0
25	0	0	0	0	0	0	0	64	0	0	0	64	0	0	0	0	64	0	0	0	64	0	64	0	192	0	64	0
26	0	0	0	0	0	0	0	64	0	0	0	0	0	0	0	64	0	64	0	64	64	0	0	64	0	192	0	0
27	0	0	0	0	0	0	0	0	64	64	0	64	0	0	0	0	0	0	0	0	64	64	0	0	64	0	192	0
28	0	0	0	0	0	0	0	0	0	64	64	64	64	0	64	64	0	0	0	0	0	0	0	0	0	0	0	192

Table A.3. The optimal SSD F constructed in Example 4.

	F_1					F_2						F_3						F_4						F_5						F_6					
Run	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	2	1	2	0	0	1	2	1	2	0	0	1	2	1	2	0	0	1	2	1	2	0	0	1	2	1	2	0	0	1	2	1	2	0
3	2	1	1	0	2	0	2	1	1	0	2	0	2	1	1	0	2	0	2	1	1	0	2	0	2	1	1	0	2	0	2	1	1	0	2
4	2	2	0	1	1	0	2	2	0	1	1	0	2	2	0	1	1	0	2	2	0	1	1	0	2	2	0	1	1	0	2	2	0	1	1
5	0	1	2	2	1	0	0	1	2	2	1	0	0	1	2	2	1	0	0	1	2	2	1	0	0	1	2	2	1	0	0	1	2	2	1
6	1	0	2	1	2	0	1	0	2	1	2	0	1	0	2	1	2	0	1	0	2	1	2	0	1	0	2	1	2	0	1	0	2	1	2
7	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2	1	1	1	1	1	1	2	2	2	2	2	2	0	0	0	0	0	0
8	1	2	1	2	0	1	2	0	2	0	1	2	0	1	0	1	2	1	2	0	2	0	1	2	0	1	0	1	2	0	1	2	1	2	0
9	2	1	1	0	2	1	0	2	2	1	0	2	1	0	0	2	1	1	0	2	2	1	0	2	1	0	0	2	1	0	2	1	1	0	2
10	2	2	0	1	1	1	0	0	1	2	2	2	1	1	2	0	0	1	0	0	1	2	2	2	1	1	2	0	0	0	2	2	0	1	1
11	0	1	2	2	1	1	1	2	0	0	2	2	2	0	1	1	0	1	1	2	0	0	2	2	2	0	1	1	0	0	0	1	2	2	1
12	1	0	2	1	2	1	2	1	0	2	0	2	0	2	1	0	1	1	2	1	0	2	0	2	0	2	1	0	1	0	1	0	2	1	2
13	0	0	0	0	0	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	2	2	2	2	2	2
14	1	2	1	2	0	2	0	1	0	1	2	1	2	0	2	0	1	1	2	0	2	0	1	0	1	2	1	2	0	2	0	1	0	1	2
15	2	1	1	0	2	2	1	0	0	2	1	1	0	2	2	1	0	1	0	2	2	1	0	0	2	1	1	0	2	2	1	0	0	2	1
16	2	2	0	1	1	2	1	1	2	0	0	1	0	0	1	2	2	1	0	0	1	2	2	0	2	2	0	1	1	2	1	1	2	0	0
17	0	1	2	2	1	2	2	0	1	1	0	1	1	2	0	0	2	1	1	2	0	0	2	0	0	1	2	2	1	2	2	0	1	1	0
18	1	0	2	1	2	2	0	2	1	0	1	1	2	1	0	2	0	1	2	1	0	2	0	0	1	0	2	1	2	2	0	2	1	0	1
19	0	0	0	0	0	2	2	2	2	2	2	2	2	2	2	2	2	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
20	1	2	1	2	0	2	0	1	0	1	2	2	0	1	0	1	2	0	1	2	1	2	0	1	2	0	2	0	1	1	2	0	2	0	1
21	2	1	1	0	2	2	1	0	0	2	1	2	1	0	0	2	1	0	2	1	1	0	2	1	0	2	2	1	0	1	0	2	2	1	0
22	2	2	0	1	1	2	1	1	2	0	0	2	1	1	2	0	0	0	2	2	0	1	1	1	0	0	1	2	2	1	0	0	1	2	2
23	0	1	2	2	1	2	2	0	1	1	0	2	2	0	1	1	0	0	0	1	2	2	1	1	1	2	0	0	2	1	1	2	0	0	2
24	1	0	2	1	2	2	0	2	1	0	1	2	0	2	1	0	1	0	1	0	2	1	2	1	2	1	0	2	0	1	2	1	0	2	0
25	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1
26	1	2	1	2	0	0	1	2	1	2	0	1	2	0	2	0	1	2	0	1	0	1	2	2	0	1	0	1	2	1	2	0	2	0	1
27	2	1	1	0	2	0	2	1	1	0	2	1	0	2	2	1	0	2	1	0	0	2	1	2	1	0	0	2	1	1	0	2	2	1	0
28	2	2	0	1	1	0	2	2	0	1	1	1	0	0	1	2	2	2	1	1	2	0	0	2	1	1	2	0	0	1	0	0	1	2	2
29	0	1	2	2	1	0	0	1	2	2	1	1	1	2	0	0	2	2	2	0	1	1	0	2	2	0	1	1	0	1	1	2	0	0	2
30	1	0	2	1	2	0	1	0	2	1	2	1	2	1	0	2	0	2	0	2	1	0	1	2	0	2	1	0	1	1	2	1	0	2	0
31	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	2	2	2	2	2	2	1	1	1	1	1	1	2	2	2	2	2	2
32	1	2	1	2	0	1	2	0	2	0	1	0	1	2	1	2	0	2	0	1	0	1	2	1	2	0	2	0	1	2	0	1	0	1	2
33	2	1	1	0	2	1	0	2	2	1	0	0	2	1	1	0	2	2	1	0	0	2	1	1	0	2	2	1	0	2	1	0	0	2	1
34	2	2	0	1	1	1	0	0	1	2	2	0	2	2	0	1	1	2	1	1	2	0	0	1	0	0	1	2	2	2	1	1	2	0	0
35	0	1	2	2	1	1	1	2	0	0	2	0	0	1	2	2	1	2	2	0	1	1	0	1	1	2	0	0	2	2	2	0	1	1	0
36	1	0	2	1	2	1	2	1	0	2	0	0	1	0	2	1	2	2	0	2	1	0	1	1	2	1	0	2	0	2	0	2	1	0	1